Interaction between the Solar Plasma Wind and the Geomagnetic Cavity

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If the quiet-time interplanetary magnetic field near the earth is parallel to the solar wind, the flow of this "collision-free" plasma over the geomagnetic cavity is reducible to an equivalent hypersonic "blunt-body problem" in ordinary gasdynamics. The shape of the converging "tail" portion of the geomagnetic cavity is determined by an equivalent gasdynamic (Prandtl-Meyer) expansion along the surface streamline. The length of the tail measured from the earth's center is about 54 earth-radii. At the aft end of the cavity, the converging plasma flow is deflected back to the axial direction across an oblique tail or "wake" shock, which decays to an Alfvén wave in a distance of some 10–20 cavity diameters downstream, or about 300 earth-radii. The geomagnetic cavity leaves a wake of hot plasma in the solar wind, and a corresponding defect in magnetic field intensity. The effects of a transverse component of the interplanetary magnetic field are illustrated by considering the flow along the stagnation line between the bow shock and the stagnation point on the magnetosphere. The axial velocity and plasma density both vanish at the stagnation point, whereas the transverse magnetic field reaches its maximum value there. This discussion furnishes the necessary first step for the analysis of the flow and magnetic field around the magnetosphere.

Nomenclature

a = sound speed, $[(\gamma kT)/m_i]^{1/2}$, cm/sec
A = Alfvén speed, $(B^2/4\pi\rho)^{1/2}$, cm/sec
$b(x)$ = lateral velocity gradient on stagnation streamline, sec^{-1}
$\mathbf{B}$ = magnetic field strength, gauss
$\mathbf{B}_E$ = intensity of undisturbed magnetic dipole field of the earth
$\mathbf{B}_G$ = intensity of geomagnetic field distorted by presence of cavity
c = speed of light, 3×10^{10} cm/sec
e = charge of proton, 4.803×10^{-10}, esu
$\mathbf{E}$ = electric field strength, emu
h, h_0 = static and total enthalpy, respectively, cm^2/sec^2
$\mathbf{j}$ = current density, emu
k = Boltzmann constant, 1.380×10^{-16} erg/°K
L = characteristic length, cm
L_c = $(2\pi v/\omega_c)$, cm
m = mass, g
M_A = Alfvén Mach number, (v/A)
n = number density, (cm^{-3})
p = pressure, dynes/cm^2
R_0 = distance from earth's center to subsolar point
R_s = bow shock radius of curvature
T = temperature, °K
u, v, w = components of velocity vector parallel to x, y, z axes, respectively, cm/sec
$\mathbf{v}$ = velocity vector, cm/sec
x, y, z = coordinate axes
γ = ratio of specific heats
ϵ = $(L_i)/(2\pi L)$
θ = ray angle measured around magnetosphere from forward stagnation point
λ = ratio of Lorentz force to inertial force $\cong 1/(M_{A_\infty}^*)^2$
μ = (m_e/m_i)
ν = collision frequency, sec^{-1}; also, turning angle in Prandtl-Meyer flow
ρ = mass density, g/cm^3
χ = angle between normal to cavity surface and direction of oncoming stream
ω_c = cyclotron frequency, $(eB)/(m_ec)$, sec^{-1}

Subscripts and Superscripts

e, i = electron, proton
2 = just behind bow shock
∞ = upstream of bow shock
$\perp$, $||$ = perpendicular and parallel to **B**, respectively
$*$ = transformed quantities defined by Eqs. (14a–14c)

1. Introduction

UNTIL recently almost all theoretical studies of the interaction between the solar plasma wind and the geomagnetic field ignored the presence of an interplanetary magnetic field.[1–5] In these theoretical models, the surface of the geomagnetic cavity consists of a thin plasma sheath in which the oncoming solar protons and electrons are deflected in opposite directions by the earth's magnetic field. This particle motion produces an electric current that annihilates the geomagnetic field everywhere outside the sheath, and approximately doubles the tangential component of the original field on the "inside edge" of the sheath.[4,5] No plasma penetrates to the interior of the cavity. The geomagnetic field on the sunlit

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side is compressed by the solar plasma wind until the magnetic pressure ($B^2/8\pi$) on the inside edge of the sheath just balances the momentum flux imparted to the solar protons by specular reflection. On this basis, the "magnetosphere" is a blunt-nosed surface on the sunlit side, slightly indented at the neutral points ($\theta \cong 70°$), with a nose radius of about 7–8 earth radii, or 5×10^4 km, and a maximum half-width of about 9×10^4 km.† In this model, the rate of closure of the "tail" portion depends upon the ratio of the thermal velocity of the protons to the plasma wind velocity. No careful study of this problem is available, but Beard[4] estimates the length of the tail to be approximately 15 times the maximum half-width of the cavity.

As a result of the Pioneer 1 and 5, Explorer 10 and 12, and especially the Mariner II experiments, there is no longer any doubt about the existence of a "quiet-time" interplanetary magnetic field in the vicinity of the earth. The mean orientation of this field is uncertain at present, but its average strength is about $10\ \gamma = 10^{-4}$ gauss. A field of this magnitude has important implications for the problem of the interaction between the solar plasma wind and the geomagnetic field. First of all, there is the question of the connection (if any) between the interplanetary and geomagnetic fields, particularly in regions of the magnetosphere where these fields are antiparallel. Secondly, the ion Larmor radius based on an average solar wind velocity of 400 km/sec is about 400 km in the interplanetary magnetic field. Since this length scale is much smaller than the estimated nose radius of the geomagnetic cavity, the interplanetary field "organizes" the gas motion somewhat as particle collisions do in ordinary gasdynamics. In addition, the solar wind velocity at the earth's location is about six times the ambient Alfvén speed, and about 15 times larger than the ambient sound speed. For these reasons, a number of investigators have suggested that the interaction between the solar plasma wind and the geomagnetic cavity is analogous to the supersonic continuum flow over a blunt-nosed body.[6–10] An important feature of this analogy (if correct) is a detached "bow shock wave" upstream of the cavity.

The purpose of this paper is two-fold: 1) to show that the proposed analogy[6–10] to supersonic continuum flow is not merely qualitative, but can be made quantitative in the special case of a purely radial interplanetary magnetic field parallel to the solar wind; and 2) to illustrate the effects of a transverse component of the interplanetary magnetic field by considering the flow along the stagnation line between the bow shock and the stagnation point (subsolar point) on the magnetosphere, including the plasma sheath. This discussion is a necessary preliminary step in the analysis of the flow and magnetic field away from the stagnation point.

In this paper we postulate the existence of a thin bow shock wave ahead of the cavity and assume that the flow is steady; both of these assumptions must be re-examined eventually for a complete understanding of this problem.

2. Flow Parameters and Equations of Motion

Except for the interior structure of shock waves and plasma sheaths, the characteristic length L governing fluid accelerations along and normal to streamlines is of the order of the thickness of the shock layer between the bow shock and the magnetosphere; i.e., $L \cong 10^4$ km. On the other hand, the important length scale for the Hall voltage and electron inertia effects is the distance $L_c = (2\pi v/\omega_c)$ that a charged particle travels along a magnetic field line during one gyration period around this line.‡ Here $\omega_c = (eB/m_c c)$ is the cyclotron frequency and $(L_c/2\pi)$ is equal to the Mach number times the usual Larmor radius.§ Suppose that the ratio of the Lorentz force to the inertial force ($\rho v^2/L$) is characterized by a parameter λ, and consider first the general case in which $\mathbf{v}$ and $\mathbf{B}$ are not everywhere parallel. Then the ratio of the "Hall voltage" term in the equation for the current[11] to the "leading term" ($\mathbf{v} \times \mathbf{B}$) is given by

$$\frac{|(c/n_e c)\mathbf{j} \times \mathbf{B}|}{|\mathbf{v} \times \mathbf{B}|} \sim \left|\left(\frac{\mathbf{v} - \mathbf{v}_e}{\mathbf{v}}\right)\right| = \left(\frac{\lambda}{2\pi}\right)\left(\frac{L_i}{L}\right) = \lambda\epsilon \quad (1)$$

where $\epsilon = L_i/(2\pi L)$.¶

A similar argument applies to the electron pressure gradient term, except that λ is replaced by a factor λ' of order unity. The ratio of the resistivity term in the equation for the current to the Hall voltage is simply (ν/ω_e), where ν is the ion-electron collision frequency. By utilizing one of Maxwell's equations

$$4\pi\mathbf{j} = \text{curl}\mathbf{B} \quad (2)$$

we find that the ratio of the leading nonlinear electron inertial term containing $(\mathbf{j}/n_e)\cdot\nabla(\mathbf{j}/n_e)$ to the Hall voltage term is given by

$$\frac{m_e c^3/e^3\,|(\mathbf{j}/n_e)\cdot\nabla(\mathbf{j}/n_e)|}{(c/n_e e)\ |\mathbf{j} \times \mathbf{B}|} \sim \frac{1}{4\pi}\left(\frac{m_e c^2}{n_e e_2}\right)\frac{1}{L^2} = \frac{\lambda}{4\pi^2}\frac{(L_i L_e)}{L^2} \quad (3)$$

This ratio is also a measure of the relative magnitude of the current density term and the Lorentz force in the momentum equation.

According to the Mariner II experiments, the magnitude of the "quiet-time" interplanetary magnetic field near the earth is about $10\gamma = 10^{-4}$ gauss, the solar wind velocity is about 400 km/sec, and $n_e \cong 10/\text{cm}^3$. Thus $\omega_i \cong 1$ rad/sec, $\omega_e \cong 1.8 \times 10^3$ rad/sec, $L_i \cong 2.5 \times 10^8$ cm, $L_e \cong 1.4 \times 10^5$ cm, and $\lambda = B_\infty^2/(4\pi\rho_\infty v_\infty^2) \cong 0.03$, and so ϵ, the ratio of the Hall voltage to ($\mathbf{v} \times \mathbf{B}$), is about 10^{-3} [Eq. (1)]; the electron pressure gradient term is about 3% of ($\mathbf{v} \times \mathbf{B}$). As expected, the electron inertia term is negligibly small, about 2×10^{-7} of the Hall voltage [Eq. (3)]. According to Spitzer,[11] in the direction parallel to the magnetic field $\nu_\| = 1.8\ (n_e \log_e\Lambda)T^{-3/2}$, whereas $\nu_\perp = 2\nu_\|$; taking $T = 5 \times 10^4$ °K and $n_e \cong 10/\text{cm}^3$, one finds $\nu_\perp \cong 8 \times 10^{-5}$/sec and $\nu_\perp/\omega_e = (\omega_e\tau_\perp)^{-1} \cong 4 \times 10^{-8}$. Thus the ohmic term is entirely negligible.

This discussion suggests that all physical quantities can be expressed as series expansions of the form

$$\begin{aligned}\mathbf{v}_i &= \mathbf{v}^{(0)} + \epsilon\mathbf{v}_i^{(1)} + \epsilon^2\mathbf{v}_i^{(2)} + \ldots \\ \mathbf{v}_e &= \mathbf{v}^{(0)} + \epsilon\mathbf{v}_e^{(1)} + \epsilon^2\mathbf{v}_e^{(2)} + \ldots\end{aligned} \quad (4)$$

etc., for $\mathbf{E}$ and $\mathbf{B}$. In this scheme,

$$\mathbf{E}^{(0)} + (\mathbf{v}^{(0)} \times \mathbf{B}^{(0)}) = 0 \quad (5)$$

This well-known form of the induction equation gives an excellent first approximation to the flow, not because the electrical conductivity is so high, but because the number density is so low and the geometric scale of events is so vast compared

† For the present, we are ignoring the inclination of the earth's axis to the plane of the ecliptic, and the earth's orbital velocity around the sun.

‡ $c = i$ for a proton; $c = e$ for an electron.

§ Following Spitzer,[11] we express the charge e/c and electric field $|\mathbf{E}|$ in electromagnetic units (emu), $|\mathbf{B}|$ in gauss, and velocity in centimeters per second.

¶ This result has a simple physical interpretation. Suppose a net acceleration $\lambda(v^2/L)$ acts in a direction $\perp$ $\mathbf{B}$. This normal acceleration produces a particle drift velocity of magnitude $\lambda(v^2/L)(1/\omega_c)$ in a direction normal to both $\mathbf{B}$ and the acceleration.[11] Since $\omega_e \gg \omega_i$, the ions drift but the electrons do not, and in the absence of particle collisions this effect appears as an electron relative velocity $|\mathbf{v} - \mathbf{v}_e| = \lambda(v^2/L)(1/\omega_i)$. Thus

$$|(\mathbf{v} - \mathbf{v}_e)/\mathbf{v}| = \lambda(L_i/2\pi L)$$

One can also show that

$$(L_i/2\pi) = 1/(\lambda)^{1/2}(m_i c^2/4\pi n_e e^2)^{1/2}$$

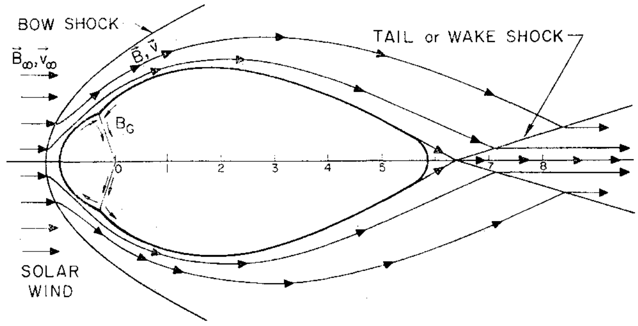

Fig. 1 Flow of solar plasma around geomagnetic cavity (radial interplanetary magnetic field). Meridian plane containing geomagnetic poles (schematic) {length unit = $R_E[4B_E^2/(40/6\pi\rho_\infty v_\infty^2)]^{1/6}$ = 9.3 earth radii}.

to laboratory plasmas.** In the special case $\mathbf{B}^{(0)} \| \mathbf{v}^{(0)}$ and $\mathbf{E}^{(0)} = 0$ everywhere (Sec. 3), the Hall voltage and the term $(\mathbf{v} \times \mathbf{B})$ are of the same order of magnitude $[0(\epsilon)]$, and must be considered together in the next approximation.

Since the flow is assumed to be steady,

$$\text{curl}\mathbf{E}^{(0)} = -\text{curl}(\mathbf{v}^{(0)} \times \mathbf{B}^{(0)}) = 0 \tag{6}$$

Also, to this same approximation, the momentum equation is (dropping superscripts)

$$\rho\mathbf{v}\cdot\nabla\mathbf{v} - (1/4\pi)\mathbf{B}\cdot\nabla\mathbf{B} = -\nabla[p + (B^2/8\pi)] \tag{7}$$

The set of governing equations is completed by Maxwell's equation [Eq. (2)] and

$$\nabla\cdot\mathbf{B} = 0 \tag{8}$$

and by the equation of continuity

$$\nabla\cdot(\rho\mathbf{v}) = 0 \tag{9}$$

3. Radial Interplanetary Magnetic Field Parallel to Solar Wind

If the interplanetary magnetic field "far" ahead of the geomagnetic cavity is parallel to the solar wind, then $\mathbf{B}\|\mathbf{v}$ throughout the flow,[13, 14]; in other words, Maxwell's "tubes of force" coincide with the streamtubes. According to Eq. (5), the magnetic flux is constant through any surface moving with the fluid[11]:

$$\int\mathbf{B}\cdot d\mathbf{A} = \text{const} \tag{10}$$

If A is taken to be the cross section of a streamtube,

$$\int\rho\mathbf{v}\cdot d\mathbf{A} = \text{const} \tag{11}$$

by conservation of mass. Thus [Eqs. (10) and (11)]

$$B = \alpha\rho\mathbf{v} \tag{12}$$

where $\alpha = (B_\infty/\rho_\infty v_\infty)$ is the same constant for all streamlines. By utilizing this relation for $\mathbf{B}$, Eq. (7) is reduced to a form identical to the momentum equation in ordinary gasdynamics,[14–16] i.e.,

$$\rho^*\mathbf{v}^*\cdot\nabla\mathbf{v}^* = -\nabla p^* \tag{13}$$

provided that

$$\mathbf{v}^* = \mathbf{v}[1 - (\alpha^2\rho/4\pi)] = \mathbf{v}[1 - (1/M_A{}^2)] \tag{14a}$$

$$\rho^*\mathbf{v}^* = \rho\mathbf{v} \text{ or } \rho^* = \rho[1 - (1/M_A{}^2)]^{-1} \tag{14b}$$

$$p^* = p + (B^2/8\pi) \tag{14c}$$

** This situation is somewhat analogous to the limiting case of low oscillation frequency compared to the ion cyclotron frequency in the propagation of small amplitude plasma waves (Lighthill[12]).

where $M_A = (v/A)$ is the Alfvén Mach number, A is the Alfvén speed, $(B^2/4\pi\rho)^{1/2}$, and [Eq. (12)]

$$M_A{}^2 = (M_A)_\infty{}^2(\rho/\rho_\infty)^{-1} \tag{15}$$

In addition [Eqs. (9) and (14b)],

$$\nabla\cdot(\rho^*\mathbf{v}^*) = 0 \tag{16}$$

The equivalent gasdynamic equations (13) and (16) for the starred quantities are to be solved subject to the boundary condition that $\mathbf{B}$, $\mathbf{v}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit vector normal to the surface of the geomagnetic cavity. Also, $\mathbf{B}_G\cdot\mathbf{n} = 0$, but $\mathbf{B}_G$ and $\mathbf{v}$ are not necessarily parallel. In addition,

$$p^* = (B_G{}^2/8\pi) \tag{17}$$

everywhere on the surface of the geomagnetic cavity; $\mathbf{B}_G$ is the geomagnetic field distorted by the presence of the cavity boundary.

According to our numerical estimates,

$$(A_\infty{}^2/a_\infty{}^2) = (B_\infty{}^2/8\pi n_i kT) \cong 6$$

where a_∞ is the ambient sound speed in the solar wind, so that $M_\infty{}^* \cong (M_A)_\infty = 5.8$. The equivalent gasdynamic flow is actually "hypersonic," and a bow "shock wave" ahead of the geomagnetic cavity is strongly suggested with these boundary conditions. At present, the mechanism for producing an irreversible change of state across a shock in this virtually collision-free plasma is not understood.[17–19] At any rate, our previous discussion (Sec. 2) suggests that the "thickness" L_S of the region over which the mean flow quantities suffer a large change must be bounded by the inequality [Eqs. (1) and (3)]

$$(1/2\pi)(L_iL_e)^{1/2} \leqq L_S \leqq (L_i/2\pi)$$

In the present problem, 10^6 cm $\leqq L_S \leqq 4 \times 10^7$ cm. So far as the mean flow quantities are concerned, the "shock wave" is at least two and possibly three orders of magnitude thinner than the "shock layer" between the geomagnetic cavity and the bow shock (Fig. 1).

Across the shock, Eq. (12) holds, and the de Hoffmann-Teller[20] conservation relations are automatically satisfied by imposing the Rankine-Hugoniot relations on the starred quantities defined by Eqs. (14a–14c). Provisionally, we take γ (ratio of specific heats) equal to 2, corresponding to the two degrees of freedom of particles in the magnetic field. The maximum density ratio $(\rho_2{}^*/\rho_\infty{}^*)$ across the shock is about 3, and $(M_A{}^2)_{\min} \cong 10$ just behind the shock [Eq. (15)]; in fact, the entire flow field is superalfvénic. By Eqs. (14a–14c), $\mathbf{v}^* = \mathbf{v}$, $\rho^* \cong \rho$ behind the shock, and

$$p^* = p[1 + (A^2/a^2)] = p\{1 + [5.4/(M_A)_\infty{}^2](v^2/v_\infty{}^2)\} \cong p\dagger\dagger$$

Thus, the principal effect of the radial interplanetary magnetic field is to prevent any significant diffusion of charged particles across streamlines and to force the central streamlines to follow the contour of the geomagnetic cavity (Fig. 1).

Numerical solutions of this equivalent hypersonic "blunt-body" problem can be obtained by the "inverse methods" of Belotserkovskii[21] and Van Dyke,[22] but a double iteration is required between the shock wave and the unknown cavity shape. At hypersonic speeds, however, the pressure on the

†† Actually there is no difficulty in supplying an equation of state for the starred quantities, if required.[14] Since the Lorentz force is everywhere normal to the streamlines, $h + (u^2/2) = h_0$, where h is the static enthalpy equal to $[\gamma/(\gamma - 1)](p/\rho)$. By introducing the usual thermodynamic relations $dh^* = (dp^*/\rho^*)$ and $a^{*2} = (\partial p^*/\partial\rho^*)$, one obtains $h^* = h + A^2[1 - \frac{1}{2}(1/M_A{}^2)]$ [using Eqs. (12) and (14c)], and

$$a^{*2} = [1 - (1/M_A{}^2)]^2\{a^2[1 - (1/M_A{}^2)] + A^2\}$$

In the present problem, $h^* \cong h + A^2$, $a^{*2} = a^2 + A^2$, and $(u^{*2}/2) + a^{*2} = h_0$.

nose of a blunt body is given fairly accurately by the modified Newtonian approximation[23]:

$$p^*/p_{\max} = \cos^2\chi \tag{18}$$

where χ is the angle between the normal to the cavity surface and the direction of the oncoming stream ahead of the bow shock. Here

$$p^*_{\max} = \rho_\infty v_\infty^2[1 - \tfrac{1}{2}(\rho_\infty^*/\rho_2^*)] = (\tfrac{5}{6})\rho_\infty v_\infty^2$$

According to Beard's approximation[4]

$$|\mathbf{B}_G| \cong 2\,|\mathbf{B}_E|_{\text{tangential}}$$

where $|\mathbf{B}_E|$ is the intensity of the undisturbed geomagnetic dipole field. When this approximation and Eq. (18) are employed in satisfying the boundary conditions on the cavity surface, the shape of the cavity in the region

$$0 \leqq \chi \leqq 60°$$

is found to be quite similar to that calculated by Spreiter and Briggs[24] for "free molecule flow," except that all their dimensions are increased by a factor of $(\frac{12}{5})^{1/6} = 1.16$. Our length unit based on the distance R_0 from the earth's center to the forward stagnation point is about 9.3 earth-radii. This cavity shape in the nose region can be regarded as the first step in an iteration procedure utilizing more accurate methods (see, for example, Hayes and Probstein[25]). Typical "profiles" across the shock layer of the magnitude of **v** and **B** in the plane containing the geomagnetic poles are shown in Fig. 2 (schematic).

The Newtonian approximation for the pressure on the cavity boundary is valid only when the normal component of the Mach number $M_\infty^* \cos\chi$ is "sufficiently large" compared to unity. Beyond about $\chi = 60°$, in our case, the pressure is governed mainly by a local gasdynamic expansion. At hypersonic speeds, this expansion is represented quite well by a Prandtl-Meyer flow around a curved surface, even for bodies of revolution.[26] The procedure is to match the Prandtl-Meyer relation[27]

$$(1/p^*)(dp^*/d\chi) = -(\gamma M^{*2})/(M^{*2} - 1)^{1/2} \tag{19}$$

with the Newtonian approximation at a point on the cavity boundary where both the pressure and pressure gradient given by these two formulas are equal.[28] Along the cavity surface

$$p^*/p^*_{\max} = (1 + \tfrac{1}{2}M^{*2})^{-2} \tag{20}$$

By employing Eqs. (18–20), we find the matching point at $\chi = 64°$, $\theta \cong 102°$, and $M^* = 1.6$. Beyond this point, M^* and χ are connected by the simple relation

$$\chi° - 64° = 57.3(\nu - 0.18) \tag{21}$$

where

$$\nu(M^*) = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left[\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2} \times (M^{*2}-1)^{1/2}\right] - \tan^{-1}(M^{*2}-1)^{1/2} \tag{22}$$

and the pressure on the cavity is obtained from Eq. (20).

Since the geomagnetic dipole field dies off rapidly with distance away from the earth, a rough estimate of the length of the tail of the geomagnetic cavity is obtained by allowing the plasma along the surface streamline to expand to vacuum. According to Eq. (22), for $\gamma = 2$ the maximum "turning angle" $\nu_{\max} = (\pi/2)\{[(\gamma+1)/(\gamma-1)]^{1/2} - 1\} = 1.15$ rad, and $\chi_{\max} = 120°$, or $\delta = \chi_{\max} - (\pi/2) = 30°$. The maximum breadth of the cavity is about 1.8 R_0 at $\tilde{x} = (x/R_0) = 2$ (Fig. 1); therefore, the *minimum* tail length is about $(2 + 1.8 \cot 30°) = 5R_0$, or about 46 earth-radii, a length comparable to the earth-moon distance. This estimate based on magnetogasdynamic considerations agrees well with Axford's[8] rough calculation based on a redistribution

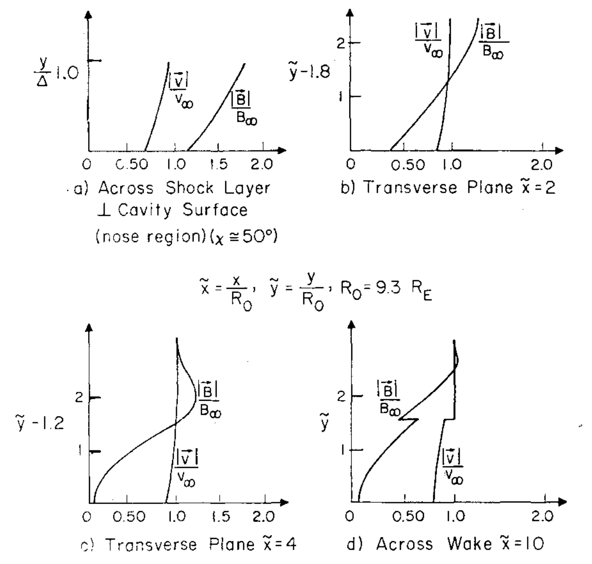

Fig. 2 Typical profiles of $|v|$ and $|B|$ (schematic).

over the tail of the total magnetic flux emerging from the polar cap, above about 70° geomagnetic latitude (neutral point). According to Fig. 1, the actual tail length is somewhat larger.

At the aft end of the cavity, the flow is deflected back to the axial direction across a magnetogasdynamic tail or "wake shock."‡‡ The estimated streamline inclination with respect to the cavity axis is about 25° just upstream of the shock, the Mach number $M^* \cong 25$, and the initial wake shock angle is about 17° with respect to the axis. The Mach number M^* just downstream of the shock is about 1.8 on the wake axis. The strength of this wake shock dies off with distance from the aft end of the cavity, and the shock approaches an Alfvén wave some 10–20 cavity diameters (300 earth-radii) downstream. Although $p^* \to p_\infty^*$ at this distance, the portion of the plasma that has passed through the "strong" portions of both bow and wake shocks is much "hotter" than the ambient plasma. We estimate that $(T^*/T_\infty^*) \cong 10$ on the wake axis "far" downstream, or $(\rho^*/\rho_\infty^*) \cong 0.10$, $v^*/v_\infty^* \cong 0.7$, and $(B/B_\infty) \cong 0.07$ (Fig. 2). In fact, in this special case of a radial interplanetary magnetic field, the wake is strikingly similar to the wake behind a blunt body at hypersonic speeds in continuum flow.[30]§§ It was this resemblance, first suggested by Leverett Davis, that drew the author's attention to the problem. Of course the main difference is that the mechanism of diffusion in the wake of the cavity is not yet understood.

4. Effect of Interplanetary Magnetic Field Transverse to Solar Plasma Wind

4.1 Statement of the Problem

Although one must be very careful in interpreting experimental data at this early stage, it seems quite probable that the interplanetary magnetic field contains a component transverse to the solar plasma wind of the same order of magnitude as the parallel component[31,32]. The existence of this component raises basic questions about the connectivity (if any) of the geomagnetic and interplanetary magnetic fields.

The most interesting situation occurs when the transverse component is antiparallel to the geomagnetic field in the

‡‡ Some remarks on tail shocks in ordinary gasdynamics can be found in Meyer.[29]

§§ The thin "boundary layer" along the cavity surface between the external plasma and the interior of the cavity may play a role in determining the location of flow separation and the strength of the wake shock.

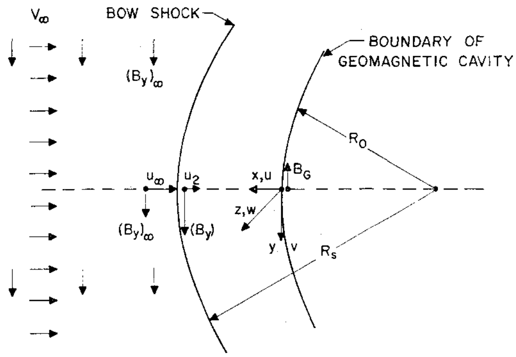

Fig. 3 Solar plasma wind with transverse magnetic field.

subsolar region. In order to explore some of the effects of this transverse magnetic field, we consider the flow along the axis of symmetry, or "stagnation line," between the bow shock and the stagnation point (Fig. 3). To simplify the discussion here, let $B_x = 0$; for an axisymmetric flow field, we put $B_z = 0$ without loss of generality. Given conditions ahead of the bow shock, all physical quantities just downstream of the shock can be calculated from the de Hoffmann-Teller[20] relations. We require $u(0) = 0$, and the location $x = x_2$ is determined as an eigenvalue. Two regions can be distinguished: 1) the "outer flow" in the shock layer, where the length scale is of order $(\rho_\infty/\rho_2)R_s$, and the approximations of Sec. 2 apply [Eqs. (6) and (7)]; and 2) a thin plasma sheath that provides a smooth transition between the antiparallel magnetic fields $B_y(0)$ and $B_G(0)$. In this zone, the approximations of Sec. 2 are clearly invalid, and one must return to the full two-fluid equations, or to the collisionless Boltzmann (Vlasov) equation itself. (There is also a third region consisting of a "buffer layer" between the inner and outer solutions.)

4.2 "Outer" Flow in Shock Layer

Since the flow ahead of the bow shock is "hyperalfvénic," the flow just behind the shock is axisymmetric when $B_x = 0$. We seek a solution that remains axisymmetric in the entire region between the shock and the boundary of the magnetosphere. Along the stagnation line,

$$v \to b(x)\cdot y \text{ and } w \to b(x)\cdot z \tag{23}$$

where the function $b(x)$ is to be determined later as part of the solution along the magnetosphere boundary away from the stagnation point. The velocity gradient $b(x)$ is of order u_∞/R_s for $\gamma = 2$.

All the other relevant physical quantities are functions of x alone to first order. With the aid of Eq. (23), the y component of the induction equation [Eq. (6)] becomes

$$d/dx(uB_y) = -b(x)\cdot B_y \tag{24}$$

or,

$$(1/B_y)(dB_y/dx) + (1/u)(du/dx) = -b(x)/u \tag{24a}$$

Along the axis, the equation of continuity [Eq. (9)] is

$$(1/\rho)(d/dx)(\rho u) + 2b(x) = 0 \tag{25}$$

By combining Eqs. (24a) and (25), we obtain the integral

$$B_y{}^2u^2 = \beta\cdot\rho u \tag{26}$$

independently of any statements about the momentum equation. Since $B_x = 0$, this relation holds also across the bow shock wave, so that

$$\beta = [u_2(B_y)_2{}^2]/\rho_2 = [u_\infty(B_y)_\infty{}^2]/\rho_\infty \tag{26a}$$

This integral shows that ρ and u approach zero simultaneously at the stagnation point, and B_y reaches its maximum value there.¶¶ This behavior is to be compared with the case of a purely radial interplanetary magnetic field (Sec. 3), where B_x and u approach zero simultaneously at the stagnation point, while ρ reaches its maximum value. The transverse magnetic field is plastered up against the boundary of the geomagnetic cavity, and the Lorentz force in the shock layer deflects the gas flow so that no plasma reaches the front side of the magnetosphere at all in the steady state. The shape of the magnetosphere is determined by the condition that the Lorentz force vanishes on the boundary, at least near the stagnation point.

By utilizing Eqs. (25), (26), and the momentum equation in the x direction [Eq. (7)], a first-order differential equation of the form $(dB_y/du) = f(u, B_y)$ is readily obtained; this equation is presently being integrated numerically. However, a simple approximate solution can be worked out if one recognizes that $u^2 \ll A^2 + a^2$ and $(\partial B_x/\partial y) \ll (\partial B_y/\partial x)$ along the stagnation line. With these approximations, the x momentum equation leads to the well-known result ($\gamma = 2$)

$$p + (B_y{}^2/8\pi) = E\rho^2 + (B_y{}^2/8\pi) = P(\text{const}) = p_2 + (B_{y_2}{}^2/8\pi) \cong (\tfrac{2}{3})\rho_\infty u_\infty{}^2 \tag{27}$$

where $E = (p_2/\rho_2{}^2)$. Then Eqs. (24, 25, and 27) give

$$(dB_y/dx) = (B_y/u)[b(x)/P][P - (B_y{}^2/8\pi)] \tag{28}$$

By utilizing the integral [Eq. (26)], we obtain the following solution after a single quadrature:

$$\int_0^x b(\xi)d\xi = \frac{S_2}{2(1-S_2)^{1/2}}(-u_2)\left[\frac{(1-S)^{1/2}}{S} + \frac{1}{2}\log_e \times \left(\frac{1+(1-S)^{1/2}}{1-(1-S)^{1/2}}\right)\right] \tag{29}$$

where [Eq. (27)]

$$S = \frac{B_y{}^2}{8\pi P} = \frac{A^2}{a^2 + A^2} \cong \frac{B_y{}^2}{\frac{16}{3}\pi\rho_\infty u_\infty{}^2} \tag{29a}$$

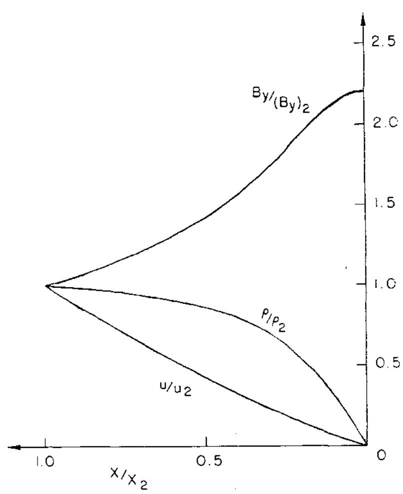

Fig. 4 Magnetic field intensity, velocity, and density along stagnation streamline.

¶¶ In a two-dimensional flow, uB_y = const [Eq. (6)], so that B_y reaches its maximum value and $\rho \to 0$ at some value of $u(x) \neq 0$, i.e., at some location ahead of the stagnation point. The present discussion shows that the rate of divergence of the velocity in the z direction in a real three-dimensional flow must be taken into account if this singularity is to be avoided. In fact, the present analysis grew out of a discussion with S. Olbert of Massachusetts Institute of Technology concerning his work on the two-dimensional problem.

In particular,

$$x_2 = -\frac{u_2}{2\bar{b}}\left[1 + \frac{1}{2}\frac{S_2}{(1 - S_2)^{1/2}}\log_e\left(\frac{1 + (1 - S_2)^{1/2}}{1 - (1 - S_2)^{1/2}}\right)\right] \quad (30)$$

where

$$x_2\bar{b} = \int_0^{x_2} b(\xi)d\xi$$

Since $S_2 \cong (1/5)$, $x_2 \cong -(u_2/2\bar{b})(1.33) = 0.22(u_\infty/\bar{b})$. We estimate that $\bar{b} \cong 0.8\ (u_\infty/R_s)$, so $(x_2/R_s) \cong 0.3$. At the stagnation point,

$$|\mathbf{B}_G| \cong 2|\mathbf{B}_E| = B_y(0)$$

so

$$(R_0/R_E) = (4B_E^2/\tfrac{16}{3}{}^*\pi\rho_\infty u_\infty^2)^{1/6} = 9.7$$

and $(R_s/R_E) \cong 14$, where R_E is the radius of the earth (6400 km). Typical distributions of B_y, ρ, and u along the axis of symmetry are shown in Fig. 4.

4.3. Plasma Sheath between Antiparallel Magnetic Fields

According to the outer solution (Sec. 4.2) $\mathbf{v}$, $\rho \to 0$ at the stagnation point, while $B_y \to B_y(0)$ and $(dB_y/dx) \to 0$ [Eq. (28)]. We require a collision-free plasma sheath separating the two regions of oppositely directed magnetic field (Fig. 5). The thickness of this sheath is several orders of magnitude smaller than the shock layer thickness, and these two magnetic fields can be regarded as uniform and plasma-free so far as the sheath structure is concerned.

A simple, exact solution of the Vlasov equation for this problem has been found by Harris.[33] He made use of the well-known fact that a solution of the Vlasov equation is given by any function of the particle constants of motion. In the plane $x = 0$, where the magnetic field vanishes (Fig. 5), both the electron and ion distribution functions are taken to be Maxwellians centered about some mean z component of velocity (W_i, W_e). By selecting $W_e = -W_i = -W$ (const), Harris transformed his reference system to one in which the electric field vanishes identically. This mean velocity W corresponds to the current generated by the deflection of the electrons moving outward from the center plane of the sheath into the magnetic field; in fact, $j_x = j_y = 0$, and

$$j_z = [(2n_e e)/c]W \quad (31)$$

Since the ion Larmor radius is so much larger than the electron Larmor radius, the ions are not deflected much in the sheath, and the resulting charge separation sets up an axial electric field that, in turn, maintains the current.* It is this field that is exactly cancelled in Harris' moving coordinate system.

Harris' results can also be obtained directly from the two-fluid equations. If we introduce the transformation

$$w = W + (E_x/B_y) \quad (32)$$

into the x component of the current equation, and utilize Eq. (31), we obtain

$$(dp_e/dx) = kT(dn_e/dx) = -(e/c)n_e W B_y \quad (33)^\dagger$$

By recognizing that

$$B_y = -(dA_z/dx) \quad (34)$$

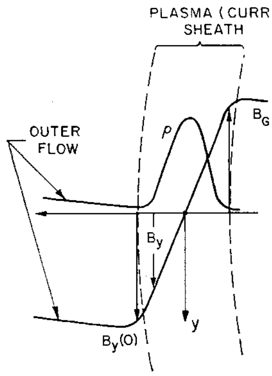

Fig. 5 Plasma sheath between geomagnetic field and magnetic shock layer.

where A_z is the z component of the vector potential, Eq. (33) integrates to

$$n_e = n_i = n_0 \exp[(eW/ckT)A_z] \quad (35)$$

Actually Harris utilized Eqs. (31), (35), and Maxwell's equation [Eq. (2)] to obtain a second-order differential equation for A_z. Alternatively, we can employ the x component of the momentum equation [Eq. (7)], which integrates to

$$p + (B_y^2/8\pi) = 2n_e kT + (B_y^2/8\pi) = (B_{y_0}^2/8\pi) \quad (36)$$

Then Eq. (31) and Maxwell's equation [Eq. (2)] lead to the first-order differential equation

$$(dB_y/dx) = (eW/2kTc)(B_{y_0}^2 - B_y^2) \quad (37)$$

the solution of which is simply

$$B_y = (16\pi n_0 kT)^{1/2}\tanh(Wx/cL_D) \quad (38)$$

where

$$(16\pi n_0 kT)^{1/2} = B_{y_0} = B_y(0) = B_G(0) \quad (38a)$$

and

$$L_D = (kT/4\pi n_0 e^2)^{1/2} \quad (38b)$$

where L_D is the debye length. If the arbitrary velocity W is identified with the electron thermal speed, then the sheath "thickness" $\sim(cL_D/W) = (m_e c^2/4\pi n_0 e^2)^{1/2}$, which is the electron Larmor radius based on the electron thermal speed and the maximum intensity of the magnetic field B_{y_0}. From Eqs. (36) and (37),

$$n_i = n_e = n = n_0[\cosh^2(Wx/cL_D)]^{-1} \quad (39)$$

The distributions of plasma density and magnetic field intensity in the sheath are indicated schematically in Fig. 5.

By taking $n_0 = 100/\text{cm}^3$, we find that the sheath thickness is about 0.5 km, as compared with a shock-layer thickness of about 2×10^4 km; thus the "buffer layer" between the plasma sheath and the outer flow is not very significant.

It should be noted that the velocity W and temperature T are arbitrary in this solution; presumably these quantities are determined by the process of sheath formation, or during the nonsteady phase associated with a solar magnetic storm.

5. Concluding Remarks

Even this preliminary analysis shows that the presence of a transverse component of the interplanetary magnetic field has a profound effect on the interaction between the solar plasma wind and the geomagnetic field. The lateral rate of divergence of the velocity in the real three-dimensional flow also plays an important role here. It certainly seems worthwhile to try to carry the present analysis around the cavity boundary away from the stagnation point, with particular attention to 1) the role played by the component of mag-

* This point seems to have been overlooked in some previous studies of similar plasma sheaths separating regions of diamagnetic plasma from regions of plasma-free magnetic field.

† Harris' solution corresponds to equal (and constant) ion and electron temperatures. Unequal temperatures can be included quite simply. (See, for example, Nicholson's[34] solution for the plasma sheath between two equal and parallel, uniform magnetic fields.)

netic field normal to the cavity boundary; and 2) the stability of the collision-free plasma sheath with respect to small disturbances.

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